## 18.152 PROBLEM SET 5

due April 29th 10:00 am.

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

**Problem 1.** *Find a solution to the following problem.*

$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < 1, t > 0 \\ &u(x,0) = g(x), u_t(x,0) = 0 && 0 \le x \le 1, \\ &u(0,t) = 0, u_x(1,t) = 0 && t \ge 0, \end{aligned}$$

*where $g$ is smooth and $g''(0) = 0$.*

Hint: Reflect $u(x,t)$ and $g(x)$ to be defined on the entire real line.

**Problem 2.** *Let $\Omega$ be an open bounded smooth domain in $\mathbb{R}^n$. Show that a smooth solution to the following Cauchy-Dirichlet problem is unique.*

$$\begin{aligned} &u_{tt} = \Delta u && x \in \Omega, t > 0 \\ &u(x,0) = g(x), u_t(x,0) = h(x) && x \in \overline{\Omega}, \\ &u(x,t) = f(x) && x \in \partial\Omega, t \ge 0, \end{aligned}$$

*where $f, g, h$ are smooth functions.*

Hint: Use the energy.

**Problem 3.** *Given smooth functions $g, h$, solve the following problem.*

$$\begin{aligned} &u_{tt} - u_{tx} - 2u_{xx} = 0 && x \in \mathbb{R}, t > 0 \\ &u(x,0) = g(x) && x \in \mathbb{R}, \\ &u_t(x,0) = h(x) && x \in \mathbb{R}. \end{aligned}$$

Hint: Use the factorization $(\partial_t - 2\partial_x)(\partial_t + \partial_x)$ and modify the d'Alembert formula.

**Problem 4.** (a) *Let $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = u_{xx} && x \in \mathbb{R}, t \ge 0 \\ &u(x,0) = g(x), u_t(x,0) = h(x) && x \in \mathbb{R}, \end{aligned}$$

*where $g, h$ are smooth. Suppose that there exists some large constant $R$ such that $g(x) = h(x) = 0$ for $|x| \geq R$. Show that there exists some large constant $T$ depending on $R$ such that*

$$(*) \qquad \int_{\mathbb{R}} |u_t(x,t)|^2 dx = \int_{\mathbb{R}} |u_x(x,t)|^2 dx,$$

*holds for $t \geq T$.*

(b) *Give an example of a solution to the wave equation on the bounded domain $(x,t) \in [0,L] \times [0,+\infty)$ such that it has constant Dirichlet data and does not satisfies* (*).

Hint: Use the d'Alembert formula.